



**University of
Zurich^{UZH}**

**Zurich Open Repository and
Archive**

University of Zurich
University Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 1996

Quaternionic monopoles

Okonek, C ; Teleman, A

Abstract: We present the simplest non-abelian version of Seiberg-Witten theory: Quaternionic monopoles. These monopoles are associated with $\text{Spin}^h(4)$ -structures on 4-manifolds and form finite-dimensional moduli spaces. On a Kähler surface the quaternionic monopole equations decouple and lead to the projective vortex equation for holomorphic pairs. This vortex equation comes from a moment map and gives rise to a new complex-geometric stability concept. The moduli spaces of quaternionic monopoles on Kähler surfaces have two closed subspaces, both naturally isomorphic with moduli spaces of canonically stable holomorphic pairs. These components intersect along a Donaldson instanton space and can be compactified with Seiberg-Witten moduli spaces. This should provide a link between the two corresponding theories.

DOI: <https://doi.org/10.1007/BF02099718>

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-22552>

Journal Article

Accepted Version

Originally published at:

Okonek, C; Teleman, A (1996). Quaternionic monopoles. *Communications in Mathematical Physics*, 180(2):363-388.

DOI: <https://doi.org/10.1007/BF02099718>

Quaternionic Monopoles

Christian Okonek*

Andrei Teleman*

Mathematisches Institut Universität Zürich
Winterthurerstrasse 190, CH-8057 Zürich

May 25, 1995

Abstract

We present the simplest non-abelian version of Seiberg-Witten theory: Quaternionic monopoles. These monopoles are associated with $Spin^h(4)$ -structures on 4-manifolds and form finite-dimensional moduli spaces. On a Kähler surface the quaternionic monopole equations decouple and lead to the projective vortex equation for holomorphic pairs. This vortex equation comes from a moment map and gives rise to a new complex-geometric stability concept. The moduli spaces of quaternionic monopoles on Kähler surfaces have two closed subspaces, both naturally isomorphic with moduli spaces of canonically stable holomorphic pairs. These components intersect along Donaldson's instanton space and can be compactified with Seiberg-Witten moduli spaces. This should provide a link between the two corresponding theories.

*Partially supported by: AGE-Algebraic Geometry in Europe, contract No ER-BCHRXCT940557 (BBW 93.0187), and by SNF, nr. 21-36111.92

0 Introduction

Recently, Seiberg and Witten [W] introduced new 4-manifold invariants, essentially by counting solutions of the monopole equations. The new invariants have already found nice applications, like e.g. in the proof of the Thom conjecture [KM] or in a short proof of the Van de Ven conjecture [OT2]. In this paper we introduce and study the simplest and the most natural non-abelian version of the Seiberg-Witten monopoles, the quaternionic monopoles.

Let (X, g) be an oriented Riemannian manifold of dimension 4. The structure group $SO(4)$ has as natural extension the quaternionic spinor group $Spin^h(4) := Spin(4) \times_{\mathbb{Z}_2} Sp(1)$:

$$1 \longrightarrow Sp(1) \longrightarrow Spin^h(4) \longrightarrow SO(4) \longrightarrow 1 .$$

The projection onto the second factor $Sp(1) = SU(2)$ induces a "determinant map" $\delta : Spin^h(4) \longrightarrow PU(2)$.

A $Spin^h(4)$ -structure on (X, g) consists of a $Spin^h(4)$ -bundle over X and an isomorphism of its $Sp(1)$ -quotient with the (oriented) orthonormal frame bundle of (X, g) . Given a $Spin^h(4)$ -structure on X , one has a one-one correspondence between $Spin^h$ -connections projecting onto the Levi-Civita connection and $PU(2)$ -connections in the associated "determinant" $PU(2)$ -bundle. The quaternionic monopole equations are:

$$\begin{cases} \not{D}_A \Psi &= 0 \\ \Gamma(F_A^+) &= (\Psi \bar{\Psi})_0 \end{cases} ,$$

where A is a $PU(2)$ -connection in the "determinant" of the $Spin^h(4)$ -structure and \not{D}_A the induced Dirac operator; Ψ is a positive quaternionic half-spinor. The Dirac operator satisfies the crucial Weitzenböck formula :

$$\not{D}_A^2 = \nabla_A^* \nabla_A + \Gamma(F_A) + \frac{s}{4} \text{id}$$

It can be used to show that the solutions of the quaternionic monopole equations are the absolute minima of a certain functional, just like in the $Spin^c(4)$ -case [JPW].

The moduli space of quaternionic monopoles associated with a fixed $Spin^h(4)$ -structure \mathfrak{h} is a real analytic space of virtual dimension

$$m_{\mathfrak{h}} = -\frac{1}{2}(3p_1 + 3e + 4\sigma) .$$

Here p_1 is the first Pontrjagin class of the determinant, e and σ denote the Euler characteristic and the signature of X .

Note that $m_{\mathfrak{h}}$ is an even integer iff X admits an almost complex structure.

The moduli spaces of quaternionic monopoles contain the Donaldson instanton moduli spaces as well as the classical Seiberg-Witten moduli spaces, which suggests that they could provide a method of comparing the two theories. We study the analytic structure around the Donaldson moduli space.

Much more can be said if the holonomy of (X, g) reduces to $U(2)$, i.e. if (X, g) is a Kähler surface. In this case we use the canonical $Spin^c(4)$ -structure with $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$ and $\Sigma^- = \Lambda^{01}$ as spinor bundles. The data of a $Spin^h(4)$ -structure \mathfrak{h} in (X, g) is then equivalent to the data of a Hermitian 2-bundle E with $\det E = \Lambda^{02}$. The determinant $\delta(\mathfrak{h})$ coincides with the $PU(2)$ -bundle $P(E)$ associated with E . A positive spinor can be written as $\Psi = \varphi + \alpha$, where $\varphi \in A^0(E^\vee)$ and $\alpha \in \Lambda^{02}(E^\vee)$ are E^\vee -valued forms. To give a $PU(2)$ -connection in $P(E)$ means to give a $U(2)$ -connection in E inducing the Chern connection in Λ^{02} , or equivalently, a $U(2)$ -connection C in E^\vee inducing the Chern connection in $K_X = \Lambda^{20}$. A pair $(C, \varphi + \alpha)$ solves the quaternionic monopole equation iff C is a connection of type $(1, 1)$, one of α or φ vanishes while the other is $\bar{\partial}_C$ -holomorphic, and a certain projective vortex equation is satisfied. This shows that in the Kähler case the moduli space decomposes as a union of two Zariski closed subspaces intersecting along the Donaldson locus. The two subspaces are interchanged by a natural real analytic involution, whose fixed point set is precisely the Donaldson moduli space.

The projective vortex equation comes from a moment map which corresponds to a new stability concept for pairs (\mathcal{E}, φ) consisting of a holomorphic bundle \mathcal{E} with canonical determinant $\det \mathcal{E} = \mathcal{K}_X$ and a holomorphic section φ . We call such a pair canonically stable iff either \mathcal{E} is stable, or $\varphi \neq 0$ and the divisorial component D_φ of the zero locus satisfies the inequality

$$c_1 \left(\mathcal{O}_X(D_\varphi)^{\otimes 2} \otimes K_X^\vee \right) \cup [\omega_g] < 0 \quad .$$

Our main result identifies the moduli spaces of irreducible quaternionic monopoles on a Kähler surface with the algebro-geometric moduli space of canonically stable pairs.

In the algebraic case, moduli spaces of quaternionic monopoles can easily be computed using our main result (Theorem 7.3) and Lemma 5.5. The moduli spaces may have several components: Every component contains a Zariski

open subset which is a holomorphic \mathbb{C}^* -bundle. For some components, this \mathbb{C}^* -bundle consists only of pairs (\mathcal{E}, φ) with \mathcal{E} stable as a bundle; components of this type can be obtained by compactifying the corresponding \mathbb{C}^* -bundle with a Donaldson moduli space at infinity. In the other direction, the component is not compact, but has a *natural compactification* obtained by adding spaces associated with Seiberg-Witten moduli spaces. The other components can also be naturally compactified by using Seiberg-Witten moduli spaces in both directions.

This compactification process, as well as the corresponding differential geometric interpretation will be the subject of a later paper.¹

1 $Spin^h$ -structures

The quaternionic spinor group is defined as

$$Spin^h := Spin \times_{\mathbb{Z}/2} Sp(1) = Spin \times_{\mathbb{Z}/2} SU(2) ,$$

and fits in the exact sequences

$$\begin{aligned} 1 &\longrightarrow Sp(1) \longrightarrow Spin^h \xrightarrow{\pi} SO \longrightarrow 1 \\ 1 &\longrightarrow Spin \longrightarrow Spin^h \xrightarrow{\delta} PU(2) \longrightarrow 1 \end{aligned} \tag{1}$$

These can be combined in the sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow Spin^h \xrightarrow{(\pi, \delta)} SO \times PU(2) \longrightarrow 1 \tag{2}$$

In dimension 4, $Spin^h(4)$ has a simple description, coming from the splitting $Spin(4) = SU(2) \times SU(2)$:

$$Spin^h(4) = SU(2) \times SU(2) \times SU(2) / \mathbb{Z}/2$$

with $\mathbb{Z}/2 = \langle (-\text{id}, -\text{id}, -\text{id}) \rangle$. There is another useful way to think of $Spin^h(4)$: let G be the group

$$G := \{(a, b, c) \in U(2) \times U(2) \times U(2) \mid \det a = \det b = \det c\} .$$

¹ After having completed our results we received a manuscript by Labastida and Marino [LM] in which related ideas are proposed from a physical point of view, and physical implications are discussed

One has an obvious isomorphism $Spin^h(4) = G/S^1$, and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 \rightarrow & \mathbb{Z}_2 & \longrightarrow & SU(2) \times SU(2) \times SU(2) & \longrightarrow & Spin^h(4) & \rightarrow 1 \\ & \downarrow & & \downarrow & & \parallel & \\ 1 \rightarrow & S^1 & \longrightarrow & G & \longrightarrow & Spin^h(4) & \rightarrow 1 \end{array} \quad (3)$$

Definition 1.1 *Let P be a principal SO -bundle over a space X . A $Spin^h$ -structure in P is a pair consisting of a $Spin^h$ bundle P^h and an isomorphism $P \simeq P^h \times_\pi SO$. The $PU(2)$ -bundle associated with a $Spin^h$ -structure is the bundle $P^h \times_\delta PU(2)$.*

Lemma 1.2 *A principal SO -bundle admits a $Spin^h$ -structure iff there exists a $PU(2)$ -bundle with the same second Stiefel-Whitney class.*

Proof: This follows from the cohomology sequence

$$\longrightarrow H^1(X, \underline{Spin^h}) \longrightarrow H^1(X, \underline{SO} \times \underline{PU(2)}) \xrightarrow{\beta} H^2(X, \mathbb{Z}/2)$$

associated to (2), since the connecting homomorphism β is given by taking the sum of the second Stiefel-Whitney classes of the two factors. ■

In this paper we will only use $Spin^h$ -structures in $SO(4)$ -bundles whose second Stiefel Whitney class admit *integral* lifts. Then we have:

Lemma 1.3 *Let P be a principal $SO(4)$ -bundle whose second Stiefel-Whitney class $w_2(P)$ is the reduction of an integral class.*

Isomorphism classes of $Spin^h(4)$ -structures in P are in 1-1 correspondence with equivalence classes of triples consisting of a $Spin^c(4)$ -structure $P^c/S^1 \simeq P$ in P , a $U(2)$ -bundle E , and an isomorphism $\det P^c \simeq \det E$, where two triples are equivalent if they can be obtained from each other by tensoring with an S^1 -bundle.

Proof: The cohomology sequence associated with the second row in (3) shows that $Spin^h$ -structures in bundles whose second Stiefel-Whitney classes

admit integral lifts are given by G -structures modulo tensoring with S^1 -bundles. On the other hand, to give a G -structure in P simply means to give a triple (Σ^+, Σ^-, E) of $U(2)$ -bundles together with isomorphisms

$$\det \Sigma^+ \simeq \det \Sigma^- \simeq \det E .$$

This is equivalent to giving a triple consisting of a $Spin^c(4)$ -structure $P^c/S^1 \simeq P$ in P , a $U(2)$ -bundle, and an isomorphism $\det P^c \simeq \det E$. ■

In the situation of this lemma, we get well defined vector bundles

$$\mathcal{H}^\pm := \Sigma^\pm \otimes E^\vee$$

depending only on the $Spin^h$ -structure and not on the chosen G -lifting. These spinor bundles have the following intrinsic interpretation: identify $SU(2) \times_{\mathbb{Z}/2}$ with $SO(4)$, and denote by

$$\pi_{ij} : Spin^h \longrightarrow SO(4)$$

the projections of $Spin^h(4) = SU(2) \times SU(2) \times SU(2)/\mathbb{Z}/2$ onto the indicated factors ($\pi = \pi_{12}$). Using the inclusion $SO(4) \subset SU(4)$, we can form three $SU(4)$ -vector bundles $P^h \times_{\pi_{ij}} \mathbb{C}^4$, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$.

Under the conditions of the previous lemma we have

$$\mathcal{H}^+ = P^h \times_{\pi_{13}} \mathbb{C}^4 , \quad \mathcal{H}^- = P^h \times_{\pi_{23}} \mathbb{C}^4 , \quad \Sigma^+ \otimes (\Sigma^-)^\vee = P^h \times_{\pi} \mathbb{C}^4 .$$

The $PU(2)$ -bundle $P^h \times_{\delta} PU(2)$ associated with the $Spin^h$ -structure $P^c/S^1 \simeq P$ has in this case a very simple description: it is the projectivization $P(E)$ of the $U(2)$ -bundle E .

2 The quaternionic monopole equations

Let (X, g) be an oriented Riemannian 4-manifold with orthonormal frame bundle P . The exact sequence (2) in the previous section shows two things: first, isomorphism classes of $PU(2)$ -bundles with second Stiefel-Whitney class equal to $w_2(P)$ are in 1-1 correspondence with orbits of $Spin^h(4)$ -structures in P under the action of $H^1(X, \mathbb{Z}/2)$; second, $Spin^h(4)$ -connections in a

$Spin^h(4)$ -bundle P^h which induce the Levi-Civita connection in P correspond bijectively to connections in the associated $PU(2)$ -bundle $P^h \times_\delta PU(2)$.

Now it is well known that $w_2(P) = w_2(X)$ is always the reduction of an integral class [HH], so that we can think of a $Spin^h$ -structure in P as a triple (Σ^+, Σ^-, E) of $U(2)$ -bundles with isomorphisms $\det \Sigma^+ \simeq \det \Sigma^- \simeq \det E$ modulo tensoring with unitary line bundles. We denote the $Spin^h(4)$ -connection corresponding to a connection $A \in \mathcal{A}(P(E))$ in the associated $PU(2)$ -bundle by \hat{A} .

Remark 2.1 *Given a fixed $U(1)$ -connection c in $\det E$, the elements in $\mathcal{A}(P(E))$ can be identified with those $U(2)$ -connections in E , which induce the fixed connection c .*

Now view a $Spin^h(4)$ -structure in P as a $Spin^c(4)$ -structure $P^c/S^1 \simeq P$ together with a $U(2)$ -bundle E and an isomorphism $\det P^c \simeq \det E$. Recall that the choice of $P^c/S^1 \simeq P$ induces an isomorphism

$$\gamma : \Lambda^1 \otimes \mathbb{C} \longrightarrow (\Sigma^+)^{\vee} \otimes \Sigma^-$$

which extends to a homomorphism

$$\Lambda^1 \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma^+ \oplus \Sigma^-) ,$$

mapping the bundle Λ^1 of real 1-forms into the bundle of trace-free skew-Hermitian endomorphisms. The induced homomorphism

$$\Gamma : \Lambda^2 \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma^+ \oplus \Sigma^-)$$

maps the subbundles $\Lambda_{\pm}^2 \otimes \mathbb{C}$ isomorphically onto the bundles $\text{End}_0(\Sigma^{\pm})$, and identifies Λ_{\pm} with the trace-free, skew-Hermitian endomorphisms ([H], [OT1]).

Definition 2.2 *Let $P^h \times_\pi SO(4) \simeq P$ be a $Spin^h(4)$ -structure in P with spinor bundle $\mathcal{H} := \mathcal{H}^+ \oplus \mathcal{H}^-$ and associated $PU(2)$ -bundle $P(E)$. Choose a connection $A \in \mathcal{A}(P(E))$, and let \hat{A} be the corresponding $Spin^h(4)$ -connection in P^h . The associated Dirac operator is defined as the composition*

$$\mathcal{D}_A : A^0(\mathcal{H}) \xrightarrow{\nabla_{\hat{A}}} A^1(\mathcal{H}) \xrightarrow{\gamma} A^0(\mathcal{H}) ,$$

where $\nabla_{\hat{A}}$ is the covariant derivative of \hat{A} and γ the Clifford multiplication.

Note that the restricted operators

$$\mathcal{D}_A : A^0(\mathcal{H}^\pm) \longrightarrow A^0(\mathcal{H}^\mp)$$

interchange the positive and negative half-spinors.

Let s be the scalar curvature of (X, g) .

Proposition 2.3 *The Dirac operator $\mathcal{D}_A : A^0(\mathcal{H}) \longrightarrow A^0(\mathcal{H})$ is an elliptic, selfadjoint operator whose Laplacian satisfies the Weitzenböck formula*

$$\mathcal{D}_A^2 = \nabla_A^* \nabla_A + \Gamma(F_A) + \frac{s}{4} \text{id}_{\mathcal{H}} \quad (4)$$

Proof: Choose a $\text{Spin}^c(4)$ -structure $P^c/S^1 \simeq P$ and a S^1 -connection c in the unitary line bundle $\det P^c$. The connection $A \in \mathcal{A}(P(E))$ lifts to a unique $U(2)$ -connection C in the bundle E^\vee which induces the dual connection of c in $\det E^\vee = (\det P^c)^\vee$. In [OT1] we introduced the Dirac operator

$$\mathcal{D}_{C,c} : A^0(\Sigma \otimes E^\vee) \longrightarrow A^0(\Sigma \otimes E^\vee) ;$$

by construction it coincides with the operator $\mathcal{D}_A : A^0(\mathcal{H}) \longrightarrow A^0(\mathcal{H})$, and its Weitzenböck formula reads

$$\mathcal{D}_{C,c}^2 = \nabla_A^* \nabla_A + \Gamma(F_{C,c}) + \frac{s}{4} \text{id}_{\mathcal{H}} ,$$

where $F_{C,c} = F_C + \frac{1}{2} F_c \text{id}_{E^\vee} \in A^2(\text{End} E^\vee)$. Substituting

$$F_C = \frac{1}{2} \text{Tr}(F_C) \text{id}_{E^\vee} + F_A$$

and using $\frac{1}{2} \text{Tr}(F_C) = -\frac{1}{2} F_c$ we get the Weitzenböck formula (4) for \mathcal{D}_A . ■

Consider now a section $\Psi \in A^0(\mathcal{H}^\pm)$. We denote by

$$(\Psi \bar{\Psi})_0 \in A^0(\text{End}_0 \Sigma^\pm \otimes \text{End}_0 E^\vee)$$

the projection of $\Psi \otimes \bar{\Psi} \in A^0(\text{End} \mathcal{H}^\pm)$ onto the fourth summand in the decomposition

$$\text{End}(\mathcal{H}^\pm) = \mathbb{C} \text{id} \oplus \text{End}_0 \Sigma^\pm \otimes \text{End}_0 E^\vee \otimes (\text{End}_0 \Sigma^\pm \otimes \text{End}_0 E^\vee) .$$

$(\Psi \bar{\Psi})_0$ is a Hermitian endomorphism which is trace-free in both factors.

Definition 2.4 Choose a $\text{Spin}^h(4)$ -structure in P with spinor bundle \mathcal{H} and associated $PU(2)$ -bundle $P(E)$. The quaternionic monopole equations for the pair $(A, \Psi) \in \mathcal{A}(P(E)) \times A^0(\mathcal{H})$ are the following equations:

$$\begin{cases} \not{D}_A \Psi &= 0 \\ \Gamma(F_A^+) &= (\Psi \bar{\Psi})_0 . \end{cases} \quad (SW^h)$$

The following result is the analog of Witten's formula in the quaternionic case (see [W], §3):

Proposition 2.5 Let $\Psi \in A^0(\mathcal{H}^+)$ be a positive half-spinor, $A \in \mathcal{A}(P(E))$ a connection in $P(E)$. Then we have

$$\begin{aligned} & \| \not{D}_A \Psi \|^2 + \frac{1}{2} \| \Gamma(F_A^+) - (\Psi \bar{\Psi})_0 \|^2 = \\ & = \| \nabla_{\hat{A}} \Psi \|^2 + \frac{1}{2} \| F_A^+ \|^2 + \frac{1}{2} \| (\Psi \bar{\Psi})_0 \|^2 + \frac{1}{4} \int_X s |\Psi|^2 . \end{aligned} \quad (5)$$

Proof: The pointwise inner product $(\Gamma(F_A) \Psi, \Psi)$ for a positive half-spinor Ψ simplifies: $(\Gamma(F_A) \Psi, \Psi) = (\Gamma(F_A^+) \Psi, \Psi) = (\Gamma(F_A^+), (\Psi \bar{\Psi})_0)$, since $\Gamma(F_A^-)$ vanishes on $A^0(\mathcal{H}^+)$, and since $\Gamma(F_A^+)$ is trace-free in both arguments.

Using the Weitzenböck formula (5), we find

$$(\not{D}_A \Psi, \Psi) = (\nabla_{\hat{A}}^* \nabla_{\hat{A}} \Psi, \Psi) + (\Gamma(F_A^+), (\Psi \bar{\Psi})_0) + \frac{s}{4} |\Psi|^2 , \quad (6)$$

which shows that

$$(\not{D}_A^2 \Psi, \Psi) + \frac{1}{2} |\Gamma(F_A^+) - (\Psi \bar{\Psi})_0|^2 = (\nabla_{\hat{A}}^* \nabla_{\hat{A}} \Psi, \Psi) + \frac{1}{2} |F_A^+|^2 + \frac{1}{2} |(\Psi \bar{\Psi})_0|^2 + \frac{s}{4} |\Psi|^2$$

The identity (5) follows by integration over X .

3 Moduli spaces of quaternionic monopoles

Let E be $U(2)$ -bundle with $w_2(P) \equiv \overline{c_1(E)} \pmod{2}$, and let c be a fixed S^1 -connection in $\det E^\vee$. We identify $\mathcal{A}(P(E))$ with the space $\mathcal{A}_c(E^\vee)$ of $U(2)$ -connections in E^\vee which induce the fixed connection in $\det E^\vee$, and we set:

$$\mathcal{A} := \mathcal{A}_c(E^\vee) \times A^0(\mathcal{H}^+)$$

The natural gauge group is the group \mathcal{G} consisting of unitary automorphisms in E^\vee which induce the identity in $\det E^\vee$. \mathcal{G} acts on \mathcal{A} from the right in a natural way. Let $\mathcal{A}^* \subset \mathcal{A}$ be the open subset of \mathcal{A} consisting of pairs (C, Ψ) whose stabilizer $\mathcal{G}_{(C, \Psi)}$ is contained in the center $\mathbb{Z}/2 = \{\pm \text{id}_E\}$ of the gauge group.

Remark 3.1 *A pair (C, Ψ) does not belong to \mathcal{A}^* iff $\Psi = 0$ and C is a reducible connection.*

Indeed, the isotropy group of \mathcal{G} acting only on the first factor $\mathcal{A}_c(E^\vee)$ is the centralizer of the holonomy of C in $SU(2)$. The latter is S^1 if C is reducible, and $\mathbb{Z}/2$ in the irreducible case. \blacksquare

A pair belonging to \mathcal{A}^* will be called irreducible. Note that the stabilizer of any pair with vanishing second component Ψ contains $\mathbb{Z}/2$.

From now on we also assume that \mathcal{A} and \mathcal{G} are completed with respect to suitable Sobolev norms L_k^2 , such that \mathcal{G} becomes a Hilbert Lie group acting smoothly on \mathcal{A} . Let $\mathcal{B} := \mathcal{A}/\mathcal{G}$ be the quotient, $\mathcal{B}^* := \mathcal{A}^*/\mathcal{G}$, and denote the orbit-map $[\] : \mathcal{A} \longrightarrow \mathcal{B}$ by π .

An element in \mathcal{A}^* will be called *strongly irreducible* if its stabilizer is trivial. Let $\mathcal{A}^{**} \subset \mathcal{A}^*$ be the subset of strongly irreducible pairs, and put $\mathcal{B}^{**} := \mathcal{A}^{**}/\mathcal{G}$.

Proposition 3.2 *\mathcal{B} is a Hausdorff space. $\mathcal{B}^{**} \subset \mathcal{B}$ is open and has the structure of a differentiable Hilbert manifold. The map $\mathcal{A}^{**} \longrightarrow \mathcal{B}^{**}$ is a differentiable principal \mathcal{G} -bundle.*

Proof: Standard, cf. [DK], [FU].

Fix a point $p = (C, \Psi) \in \mathcal{A}$. The differential of the map $\mathcal{G} \longrightarrow \mathcal{A}$ given by the action of \mathcal{G} on p is the map

$$\begin{array}{ccc} D_p^0 : A^0(\text{su}(E^\vee)) & \longrightarrow & A^1(\text{su}(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \\ f & \longmapsto & (D_C(f), -f\Psi) \end{array}$$

Setting

$$N_p(\varepsilon) := \{ \beta \in A^1(\text{su}(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \mid D_p^{0*} \beta = 0, \|\beta\| < \varepsilon \} ,$$

for $\varepsilon > 0$ sufficiently small, one obtains local slices for the action of \mathcal{G} on \mathcal{A}^{**} and charts $\pi|_{N_p(\varepsilon)} : N_p(\varepsilon) \longrightarrow \mathcal{B}^{**}$ for \mathcal{B}^{**} . \blacksquare

Note that the curvature F_A of a connection in $P(E)$ equals the trace-free part F_C^0 of the curvature of the corresponding connection $C \in \mathcal{A}_c(E^\vee)$.

Using the identification $\mathcal{A}(P(E)) = \mathcal{A}_c(E^\vee)$, we can rewrite the quaternionic monopole equations in terms of pairs $(C, \Psi) \in \mathcal{A}$. Let $\mathcal{A}^{SW^h} \subset \mathcal{A}$ be the space of solutions.

Definition 3.3 *Fix a $Spin^h$ -structure in P . The moduli space of quaternionic monopoles is the quotient $\mathcal{M} := \mathcal{A}^{SW^h}/\mathcal{G}$. We denote by $\mathcal{M}^{**} := (\mathcal{A}^{SW^h} \cap \mathcal{A}^{**})/\mathcal{G}$, $\mathcal{M}^* := (\mathcal{A}^{SW^h} \cap \mathcal{A}^*)/\mathcal{G}$ the subspaces of (strongly) irreducible monopoles.*

The tangent space to \mathcal{A}^{SW^h} at $p = (C, \Psi) \in \mathcal{A}$ is the kernel of the operator

$$D_p^1 : A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \longrightarrow A^0(su(\Sigma^+) \otimes su(E^\vee)) \oplus A^0(\Sigma^- \otimes E^\vee)$$

defined by

$$D_p^1((\alpha, \psi)) = \left(\Gamma(D_C^+(\alpha)) - [(\psi\bar{\Psi})_0 + (\Psi\bar{\psi})_0], \mathcal{D}_{C,c}\psi + \gamma(\alpha)\Psi \right),$$

where we consider $\gamma(\alpha)$ as map $\gamma(\alpha) : \Sigma^+ \longrightarrow \Sigma^- \otimes su(E^\vee)$. Clearly $D_p^1 \circ D_p^0 = 0$, since the monopole equations are gauge invariant.

Using the isomorphism $\Gamma^{-1} : A^0(su(\Sigma^+)) \longrightarrow A_+^2$, we can consider D_p^1 as an operator $D_p^1 : A^1(su(E^\vee)) \oplus A^0(\Sigma^+ \otimes E^\vee) \longrightarrow A_+^2(su(E^\vee)) \oplus A^0(\Sigma^- \otimes E^\vee)$.

Let $\sigma(X)$ and $e(X)$ be the signature and the topological Euler characteristic of the oriented manifold X .

Proposition 3.4 *For a solution $p = (C, \Psi) \in \mathcal{A}^{SW^h}$, the complex*

$$0 \rightarrow A^0 su(E^\vee) \xrightarrow{D_p^0} A^1 su(E^\vee) \oplus A^0 \mathcal{H}^+ \xrightarrow{D_p^1} A_+^2 su(E^\vee) \oplus A^0 \mathcal{H}^- \rightarrow 0 \quad (\mathcal{C}_p)$$

is elliptic and its index is

$$\frac{3}{2}(4c_2(E^\vee) - c_1(E^\vee)^2) - \frac{1}{2}(3e(X) + 4\sigma(X)). \quad (7)$$

Proof: The complex \mathcal{C}_p has the same symbol sequence as

$$0 \rightarrow A^0 su(E^\vee) \xrightarrow{(D_C, 0)} A^1 su(E^\vee) \oplus A^0 \mathcal{H}^+ \xrightarrow{(D_C^+, \mathcal{P}_{C,c})} A_+^2 su(E^\vee) \oplus A^0 \mathcal{H}^- \rightarrow 0$$

which is an elliptic complex with index

$$2(4c_2(E^\vee) - c_1(E^\vee)^2) - \frac{3}{2}(\sigma(X) + e(X)) + \text{index } \mathcal{P}_{C,c}.$$

The latter term is

$$\text{index } \mathcal{P}_{C,c} = [ch(E^\vee) e^{\frac{1}{2}c_1(E^\vee)} \hat{A}(X)]_4 = -2c_2(E^\vee) + \frac{1}{2}c_1(E^\vee)^2 - \frac{1}{2}\sigma(X).$$

■

Remark 3.5 *The integer in (7) is always an even number if X admits almost complex structures.*

Our next step is to endow the spaces \mathcal{M}^{**} (\mathcal{M}^*) with the structure of a real analytic space (orbifold).

In the first case (compare with [FU], [DK], [OT1], [LT]), we have an analytic map $\sigma : \mathcal{A} \rightarrow A_+^2(su(E^\vee)) \oplus A^0(\mathcal{H}^-)$ defined by

$$\sigma(C, \Psi) = \left((F_C^0)^+ - \Gamma^{-1}(\Psi \bar{\Psi})_0, \mathcal{P}_{C,c} \Psi \right)$$

which gives rise to a section $\tilde{\sigma}$ in the bundle $\mathcal{A}^{**} \times_{\mathcal{G}} (A_+^2(su(E^\vee)) \oplus A^0(\mathcal{H}^-))$. We endow \mathcal{M}^{**} with a real analytic structure by identifying it with the vanishing locus $Z(\tilde{\sigma})$ of $\tilde{\sigma}$, regarded as a subspace of the Hilbert manifold \mathcal{B}^{**} (in Douady's sense) ([M], [LT]).

Now fix a point $p = (C, \Psi) \in \mathcal{A}^*$. We put

$$S_p(\varepsilon) := \{p + \beta \mid \beta \in A^1 su(E^\vee) \oplus A^0 \mathcal{H}^+, D_p^0 D_p^{0*} \beta + D_p^{1*} \sigma(p + \beta) = 0, \|\beta\| < \varepsilon\}.$$

Claim 3.6 *For sufficiently small $\varepsilon > 0$, $S_p(\varepsilon)$ is a finite dimensional submanifold of \mathcal{A} which is contained in the slice $N_p(\varepsilon)$ and whose tangent space at p is the first harmonic space \mathbb{H}_p^1 of the deformation complex \mathcal{C}_p .*

To prove this claim, we consider the map

$$s_p : A^1(su(E^\vee)) \oplus A^0(\mathcal{H}^+) \longrightarrow \text{im}(D_p^0) \oplus \text{im}(D_p^1)^*$$

given by the left hand terms in the equations defining $S_p(\varepsilon)$. The derivative of s_p at 0 is the first Laplacian

$$\Delta_p^1 : A^1(su(E^\vee)) \oplus A^0(\mathcal{H}^+) \longrightarrow \text{im}(D_p^0) \oplus \text{im}(D_p^1)^*$$

associated with the elliptic complex \mathcal{C}_p , hence s_p is a submersion in 0. This proves the claim. \blacksquare

The intersection $\mathcal{A}^{SW^h} \cap N_p(\varepsilon) = Z(\sigma) \cap N_p(\varepsilon)$ of the space of solutions with the standard slice through p is contained in $S_p(\varepsilon)$ and can be identified with the finite dimensional model

$$Z(\sigma) \cap N_p(\varepsilon) = Z(\sigma|_{S_p(\varepsilon)}) .$$

If $p \in \mathcal{A}^{**}$ is strongly irreducible, then the map

$$\pi|_{Z(\sigma|_{S_p(\varepsilon)})} : Z(\sigma|_{S_p(\varepsilon)}) \longrightarrow \mathcal{M}^{**}$$

is a local parametrization of \mathcal{M}^* at p , hence $Z(\sigma|_{S_p(\varepsilon)})$ is a local model for the moduli space around p .

If $p \in \mathcal{A}^* \setminus \mathcal{A}^{**}$ is irreducible but not strongly irreducible, then necessarily $\Psi = 0$, and the isotropy group $\mathcal{G}_p = \mathbb{Z}/2$ acts on $S_p(\varepsilon)$. Since σ is $\mathbb{Z}/2$ -equivariant, we obtain an induced action on $Z(\sigma|_{S_p(\varepsilon)})$. In this case $\pi|_{Z(\sigma|_{S_p(\varepsilon)})}$ induces a homeomorphism of the quotient $Z(\sigma|_{S_p(\varepsilon)})/\mathbb{Z}/2$ with an open neighbourhood of p in \mathcal{M}^* , and \mathcal{M}^* becomes an orbifold at p , if we use the map

$$\pi|_{Z(\sigma|_{S_p})} : Z(\sigma|_{S_p(\varepsilon)}) \longrightarrow \mathcal{M}^*$$

as an orbifold chart.

Remark 3.7 *Using a real analytic isomorphism which identifies the germ of $S_p(\varepsilon)$ at p with the germ of $\mathbb{H}_p^1 = T_p(S_p(\varepsilon))$ at 0, we obtain a local model of Kuranishi-type for \mathcal{M}^* at p .*

Remark 3.8 *The points in $\mathcal{D}^* := \mathcal{M}^* \setminus \mathcal{M}^{**}$ have the form $[(C, 0)]$, where C is projectively anti-self-dual, i.e. $(F_C^0)^+ = 0$. There is a natural finite map*

$$\mathcal{D}^* \longrightarrow \mathcal{M}(P(E^\vee))$$

into the Donaldson moduli space of $PU(2)$ -instantons in $P(E^\vee)$, which maps \mathcal{D}^ isomorphically onto $\mathcal{M}(P(E^\vee)^*)$ if $H^1(X, \mathbb{Z}/2) = 0$. In general \mathcal{D}^* and $\mathcal{M}(P(E^\vee)^*)$ cannot be identified. The difference comes from the fact that our gauge group is $SU(E^\vee)$, whereas the $PU(2)$ -instantons are classified modulo $PU(E^\vee)$.*

For simplicity we shall however refer to \mathcal{D}^* as Donaldson instanton moduli space.

Concluding, we get

Proposition 3.9 *\mathcal{M}^{**} is a real analytic space. \mathcal{M}^* is a real analytic orbifold, and the points in $\mathcal{M}^* \setminus \mathcal{M}^{**}$ have neighbourhoods modeled on $\mathbb{Z}/2$ -quotients. $\mathcal{M}^* \setminus \mathcal{M}^{**}$ can be identified as a set with the Donaldson moduli space \mathcal{D}^* of irreducible projectively anti-self-dual connections in E^\vee with fixed determinant c .*

The local structure of the moduli space \mathcal{M} in reducible points, which correspond to pairs formed by a reducible instanton and a trivial spinor, can also be described using the method above (compare with [DK]).

Let $\mathcal{M}^{SW} \subset \mathcal{M}$ be the subspace of \mathcal{M} consisting of all orbits of the form $(C, \Psi) \cdot SU(E^\vee)$, where C is a reducible connection and Ψ belongs to one of the summands. Let $L := \det \Sigma^\pm = \det E$. It is easy to see that

$$\mathcal{M}^{SW} \simeq \bigcup_{\substack{S \text{ summand} \\ \text{of } E^\vee}} \mathcal{M}_{L \otimes S^{\otimes 2}}^{SW},$$

where \mathcal{M}_M^{SW} denotes the rank-1 Seiberg-Witten moduli space associated to a $Spin^c(4)$ -structure of determinant M .

The fact that the moduli spaces of quaternionic monopoles contain Donaldson moduli spaces as well of Seiberg-Witten moduli spaces suggests that they could provide a method for comparing the invariants given by the two theories.

4 Quaternionic monopoles on Kähler surfaces

Let (X, g) be a Kähler surface with canonical $Spin^c(4)$ -structure; in this case $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$, and $\Sigma^- = \Lambda^{01}$. A $Spin^h(4)$ -structure in the frame bundle is given by a unitary vector bundle E together with an isomorphism $\det E \simeq \Lambda^{02}$. A $Spin^h(4)$ -connection \hat{A} corresponds to a $PU(2)$ -connection A in the associated bundle $P(E)$, or alternatively, to a unitary connection C in E^\vee which induces a fixed S^1 -connection c in Λ^{20} . Recall that the curvature F_A of A equals the trace-free component F_C^0 of F_C .

If we choose c to be the Chern connection in the canonical bundle $\Lambda^{20} = K_X$, then the $Spin^h(4)$ -connection in $\mathcal{H} = \Sigma \otimes E^\vee$ is simply the tensor product of the canonical connection in $\Sigma = \Sigma^+ \oplus \Sigma^-$ and the connection C .

A positive quaternionic spinor $\Psi \in A^0(\mathcal{H}^+)$ can be written as $\Psi = \varphi + \alpha$, with $\varphi \in A^0(E^\vee)$, and $\alpha \in A^{02}(E^\vee)$.

Proposition 4.1 *Let C be a unitary connection in E^\vee inducing the Chern connection c in $\det E^\vee = K_X$. A pair $(C, \varphi + \alpha)$ solves the quaternionic monopole equations if and only if F_C is of type $(1, 1)$ and one of the following conditions holds*

$$\begin{aligned} 1. & \alpha = 0, \quad \bar{\partial}_C \varphi = 0 \text{ and } i\Lambda_g F_C^0 + \frac{1}{2}(\varphi \otimes \bar{\varphi})_0 = 0, \\ 2. & \varphi = 0, \quad \partial_C \alpha = 0 \text{ and } i\Lambda_g F_C^0 - \frac{1}{2} * (\alpha \otimes \bar{\alpha})_0 = 0. \end{aligned} \tag{8}$$

Proof: Using the notation in the proof of the Weitzenböck formula, we have $F_{C,c} = \frac{1}{2}(\text{Tr} F_C + F_c)\text{id}_{E^\vee} + F_A = F_A = F_C^0 \in A^2(su(E^\vee))$. By Proposition 2.6 of [OT1] the quaternionic Seiberg-Witten equations become

$$\begin{cases} F_A^{20} &= -\frac{1}{2}(\varphi \otimes \bar{\alpha})_0 \\ F_A^{02} &= \frac{1}{2}(\alpha \otimes \bar{\varphi})_0 \\ i\Lambda_g F_A &= -\frac{1}{2}[(\varphi \otimes \bar{\varphi})_0 - *(\alpha \otimes \bar{\alpha})_0] \\ \bar{\partial}_C \varphi &= i\Lambda_g \partial_C \alpha \end{cases}.$$

Note that the right-hand side of formula (5) is invariant under Witten's transformation $(C, \varphi + \alpha) \mapsto (C, \varphi - \alpha)$. Therefore, every solution satisfies $F_A^{20} = F_A^{02} = 0$, and $(\varphi \otimes \bar{\alpha})_0 = (\alpha \otimes \bar{\varphi})_0 = 0$. Elementary computations show that this can happen only if $\varphi = 0$ or $\alpha = 0$. On the other hand, since the Chern connection in K_X is integrable, we also get $F_C^{20} = F_C^{02} = 0$. ■

Remark 4.2 *The second case in this proposition reduces to the first: in fact, if $\varphi = 0$ and $\alpha \in A^{02}(E^\vee)$ satisfies $i\Lambda_g \partial \alpha = 0$, we set $\psi := \bar{\alpha} \in A^{20}(\bar{E}^\vee) = A^0(\Lambda^{20} \otimes E) = A^0(E^\vee)$, and we obtain $\bar{\partial}_C \psi = \overline{\partial_C \psi} = \overline{\partial_C \alpha} = 0$. Here we used the fact that $\Lambda_g : \Lambda^{12} \longrightarrow \Lambda^{01}$ is an isomorphism, the adjoint of the Lefschetz isomorphism $\cdot \wedge \omega_g$ [LT]. A simple calculation in coordinates gives $- * (\alpha \otimes \bar{\alpha})_0 = (\bar{\alpha} \otimes \bar{\alpha})_0 = (\psi \otimes \bar{\psi})_0$.*

5 Stability

Let (X, g) be a compact Kähler manifold of arbitrary dimension, E a differentiable vector bundle, and let \mathcal{L} be a fixed holomorphic line bundle, whose underlying differentiable line bundle is $L := \det E$.

Definition 5.1 *A holomorphic pair of type (E, \mathcal{L}) is a pair (\mathcal{E}, φ) consisting of a holomorphic bundle \mathcal{E} and a section $\varphi \in H^0(X, \mathcal{E})$ such that the underlying differentiable bundle of \mathcal{E} is E and $\det \mathcal{E} = \mathcal{L}$.*

Note that the determinant of the holomorphic bundle \mathcal{E} is fixed, not only its isomorphism type.

Two pairs $(\mathcal{E}_i, \varphi_i)$, $i = 1, 2$ of the same type are isomorphic if there exists an isomorphism $f : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with $f^*(\varphi_2) = \varphi_1$ and $\det f = \text{id}_{\mathcal{L}}$.

In other words, $(\mathcal{E}_i, \varphi_i)$ are isomorphic iff there exists a complex gauge transformation $f \in SL(E)$ with $f^*(\varphi_2) = \varphi_1$ such that f is holomorphic as a map $f : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$.

Definition 5.2 *A holomorphic pair (\mathcal{E}, φ) is simple if any automorphism of it is of the form $f = \varepsilon \text{id}_{\mathcal{E}}$, where $\varepsilon^{\text{rk} \mathcal{E}} = 1$. A pair (\mathcal{E}, φ) is strongly simple if its only automorphism is $\text{id}_{\mathcal{E}}$.*

Note that a simple pair (\mathcal{E}, φ) with $\varphi \neq 0$ is strongly simple, whereas a pair $(\mathcal{E}, 0)$ is simple iff \mathcal{E} is a simple bundle.

Note also that (\mathcal{E}, φ) is simple iff any trace-free holomorphic endomorphism f of \mathcal{E} with $f(\varphi) = 0$ vanishes.

For a nontrivial torsion free sheaf \mathcal{F} on X , we denote by $\mu_g(\mathcal{F})$ its slope with respect to the Kähler metric g . Given a holomorphic bundle \mathcal{E} over X and a holomorphic section $\varphi \in H^0(X, \mathcal{E})$, we let $\mathcal{S}(\mathcal{E})$ be the set of reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$, and we define

$$\mathcal{S}_\varphi(\mathcal{E}) := \{\mathcal{F} \in \mathcal{S}(\mathcal{E}) \mid \varphi \in H^0(X, \mathcal{F})\} .$$

Recall the following stability concepts [B2]:

Definition 5.3

1. \mathcal{E} is φ -stable if

$$\max \left(\mu_g(\mathcal{E}), \sup_{\mathcal{F}' \in \mathcal{S}(\mathcal{E})} \mu_g(\mathcal{F}') \right) < \inf_{\mathcal{F} \in \mathcal{S}_\varphi(\mathcal{E})} \mu_g(\mathcal{E}/\mathcal{F}) .$$

2. Let $\lambda \in \mathbb{R}$ be a real parameter. The pair (\mathcal{E}, φ) is λ -stable iff

$$\max \left(\mu_g(\mathcal{E}), \sup_{\mathcal{F}' \in \mathcal{S}(\mathcal{E})} \mu_g(\mathcal{F}') \right) < \lambda < \inf_{\mathcal{F} \in \mathcal{S}_\varphi(\mathcal{E})} \mu_g(\mathcal{E}/\mathcal{F}) .$$

3. (\mathcal{E}, φ) is called λ -polystable if \mathcal{E} splits holomorphically as $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, such that $\varphi \in H^0(X, \mathcal{E}')$, (\mathcal{E}', φ) is a λ -stable pair, and \mathcal{E}'' is a polystable vector bundle of slope λ .

From now on we restrict ourselves to the case $\text{rk}(\mathcal{E}) = 2$.

Definition 5.4

1. A holomorphic pair (\mathcal{E}, φ) of type (E, \mathcal{L}) is called stable if one of the following conditions is satisfied:

i) \mathcal{E} is φ -stable.

ii) $\varphi \neq 0$ and \mathcal{E} splits in direct sum of line bundle $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, such that $\varphi \in H^0(\mathcal{E}')$ and the pair (\mathcal{E}', φ) is $\mu_g(E)$ -stable.

2. A holomorphic pair (\mathcal{E}, φ) of type (E, \mathcal{L}) is called polystable if it is stable, or $\varphi = 0$ and \mathcal{E} is a polystable bundle.

Note that there is no parameter λ in the stability concept for holomorphic pairs of a fixed type. The conditions depend only on the metric g and on the slope $\mu_g(E)$ of the underlying differentiable bundle E .

Lemma 5.5 Let (\mathcal{E}, φ) be a holomorphic pair of type (E, \mathcal{L}) with $\varphi \neq 0$. There exists a uniquely determined effective divisor $D = D_\varphi$ and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(D) & \xrightarrow{\varphi} & \mathcal{E} & \longrightarrow & \mathcal{L}(-D) \otimes J_Z \longrightarrow 0 , \\ & & \uparrow D & & \nearrow \varphi & & \\ & & \mathcal{O}_X & & & & \end{array} \quad (9)$$

with a local complete intersection $Z \subset X$ of codimension 2. The pair (\mathcal{E}, φ) is stable if and only if $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$.

Proof: $D = D_\varphi$ is the divisorial component of the zero locus $Z(\varphi)$ of \mathcal{E} which is defined by the ideal $\text{im}(\varphi^\vee : \mathcal{E}^\vee \longrightarrow \mathcal{O}_X)$, and $\hat{\varphi}$ is the induced map. The set $\mathcal{S}_\varphi(\mathcal{E})$ consists precisely of the line bundles $\mathcal{F} \subset \mathcal{O}_X(D)$, so that

$$\inf_{\mathcal{F} \in \mathcal{S}_\varphi(\mathcal{E})} \mu_g(\mathcal{E}/\mathcal{F}) = 2\mu_g(E) - \mu_g(\mathcal{O}_X(D)) .$$

Suppose (\mathcal{E}, φ) is stable. If \mathcal{E} is φ -stable, we have $\mu_g(E) < 2\mu_g(E) - \mu_g(\mathcal{O}_X(D))$, which gives the required inequality. If \mathcal{E} is not φ -stable, then $Z = \emptyset$, the extension (9) splits, and the pair $(\mathcal{O}_X(D), \varphi)$ is $\mu_g(E)$ -stable, i.e. $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$.

Conversely, suppose $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$, and assume first that the extension (9) does not split. In this case \mathcal{E} is φ -stable: in fact, if $\mathcal{F}' \subset \mathcal{E}$ is an arbitrary line bundle, either $\mathcal{F}' \subset \mathcal{O}_X(D)$, or the induced map $\mathcal{F}' \subset \mathcal{E} \longrightarrow \mathcal{J}_Z \otimes \mathcal{L}(-D)$ is non-trivial. But then $\mathcal{F}' \simeq \mathcal{L} \otimes \mathcal{O}_X(-D - \Delta)$ for an effective divisor Δ containing Z , and we find

$$\mu_g(\mathcal{F}') = 2\mu_g(E) - \mu_g(D) - \mu_g(\Delta) \leq 2\mu_g(E) - \mu_g(\mathcal{O}_X(D)) .$$

Furthermore, strict inequality holds, unless $Z = \emptyset$ and the extension (9) splits, which it does not by assumption.

In the case of a split extension, we only have to notice that a pair (\mathcal{E}', φ) is λ -stable for any parameter $\lambda > \mu_g(\mathcal{E}')$ [B1]. \blacksquare

Remark 5.6 Consider a pair (\mathcal{E}, φ) of type (E, \mathcal{L}) with $\varphi \neq 0$ and associated extension (9). \mathcal{E} is φ -stable iff $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$, and the extension does not split.

Indeed, if the extension splits, then \mathcal{E} is not φ -stable, since

$$\mu_g(\mathcal{L}(-D)) = \inf_{\mathcal{F} \in \mathcal{S}_\varphi(\mathcal{E})} \mu_g(\mathcal{E}/\mathcal{F}) .$$

6 The projective vortex equation

Let E be a differentiable vector bundle over a compact Kähler manifold (X, g) . We fix a holomorphic line bundle \mathcal{L} and a Hermitian metric l in \mathcal{L} . Let (\mathcal{E}, φ) be a holomorphic pair of type (E, \mathcal{L}) .

Definition 6.1 A Hermitian metric in \mathcal{E} with $\det h = l$ is a solution of the projective vortex equation iff the trace free part F_h^0 of the curvature F_h satisfies the equation

$$i\Lambda_g F_h^0 + \frac{1}{2}(\varphi\bar{\varphi}^h)_0 = 0 . \quad (V)$$

Theorem 6.2 Let (\mathcal{E}, φ) be a holomorphic pair of type (E, \mathcal{L}) with $\text{rk}(\mathcal{E}) = 2$. Fix a Hermitian metric l in \mathcal{L} .

The pair (\mathcal{E}, φ) is polystable iff \mathcal{E} admits a Hermitian metric h with $\det h = l$ which is a solution of the projective vortex equation. If (\mathcal{E}, φ) is stable, then the metric h is unique.

Proof: Suppose first that h is a solution of the projective vortex equation (V). Then we have

$$i\Lambda F_h + \frac{1}{2}(\varphi\bar{\varphi}^h) = \frac{1}{2}(i\Lambda \text{Tr} F_h + \frac{1}{2}|\varphi|^2)\text{id}_E ,$$

i.e. h satisfies the weak vortex equation (V_t) associated to the real function $t := \frac{1}{2}(2i\Lambda \text{Tr} F_h + |\varphi|^2)$. Therefore, by [OT1], the pair (\mathcal{E}, φ) is λ -polystable for the parameter $\lambda = \frac{(n-1)!}{4\pi} \int_X \text{vol}_g = \mu_g(\mathcal{E}) + \frac{(n-1)!}{8\pi} \|\varphi\|^2$.

Let A be the Chern connection of h , and denote by \mathcal{E}' the minimal A -invariant subbundle which contains φ . If $\mathcal{E}' = \mathcal{E}$, then \mathcal{E} is φ -stable and the pair (\mathcal{E}, φ) is stable.

If $\mathcal{E}' = 0$, hence $\varphi = 0$, then h is a weak Hermitian-Einstein metric, \mathcal{E} is a polystable bundle, and the pair (\mathcal{E}, φ) is polystable by definition.

In the remaining case \mathcal{E}' is a line bundle and $\varphi \neq 0$. Let $\mathcal{E}'' := \mathcal{E}'^\perp$ be the orthogonal complement of \mathcal{E}' , and let h' and h'' be the induced metrics in \mathcal{E}' and \mathcal{E}'' . We put $s := i\Lambda_g \text{Tr} F_h$. Then, since $h = h' \oplus h''$, the projective vortex equation can be rewritten as:

$$\begin{cases} i\Lambda F_{h'} + \frac{1}{2}(\varphi\bar{\varphi}^{h'}) = \frac{1}{2}(s + \frac{1}{2}|\varphi|_{h'}^2)\text{id}_{\mathcal{E}'} \\ i\Lambda F_{h''} = \frac{1}{2}(s + \frac{1}{2}|\varphi|_{h'}^2)\text{id}_{\mathcal{E}''} . \end{cases}$$

The first of these equations is equivalent to

$$i\Lambda F_{h'} + \frac{1}{4}(\varphi\bar{\varphi}^{h'}) = \frac{s}{2}\text{id}_{\mathcal{E}'} ,$$

which implies that $(\mathcal{E}', \frac{\varphi}{\sqrt{2}})$ is $\mu_g(\mathcal{E})$ -stable by [OT1].

Conversely, suppose first that (\mathcal{E}, φ) is stable. We have to consider two cases:

Case 1: \mathcal{E} is φ stable.

Using Bradlow's existence theorem, we obtain Hermitian metrics in \mathcal{E} satisfying the usual vortex equations associated with suitable chosen λ , and, of course these metrics all satisfy the equation (V). The problem is, however, to find a solution with an a priori given determinant l .

In order to achieve this stronger result, Bradlow's proof has to be modified slightly at some points:

One starts by fixing a background metric k such that $\det k = l$. Denote by $S_0(k)$ the space of trace-free k -Hermitian endomorphisms of E , and let $\mathcal{M}et(l)$ be the space of Hermitian metrics in E with $\det h = l$. On

$$\mathcal{M}et(l)_2^p := \{ke^s \mid s \in L_2^p(S_0(k))\}$$

we define the functional $M_\varphi : \mathcal{M}et(l)_2^p \longrightarrow \mathbb{R}$ by

$$M_\varphi(h) := M_D(k, h) + \|\varphi\|_h^2 - \|\varphi\|_k^2.$$

Here M_D is the Donaldson functional, which is known to satisfy the identity $\frac{d}{dt}M_D(k, h(t)) = 2 \int_X \text{Tr}[h^{-1}(t)\dot{h}(t)i\Lambda_g F_h]$ for any smooth path of metrics $h(t)$ [Do], [Ko]. Since $h^{-1}(t)\dot{h}(t)$ is trace-free for a path in $\mathcal{M}et(l)$, we obtain

$$\frac{d}{dt}M_D(k, h(t)) = 2 \int_X \text{Tr}[h^{-1}\dot{h}(t)i\Lambda_g F_h^0].$$

Similarly, for a path of the form $h(t) = he^{ts}$, with $s \in S_0(h)$, we get

$$\frac{d}{dt}\bigg|_{t=0} \|\varphi\|_{h_t}^2 = \frac{d}{dt}\bigg|_{t=0} \langle e^{ts}\varphi, \varphi \rangle_h = \left\langle \frac{d}{dt}\bigg|_{t=0} e^{ts}\varphi, \varphi \right\rangle_h = \langle s, \varphi \bar{\varphi}^h \rangle_h = \int_X \text{Tr}[s(\varphi \bar{\varphi}^h)_0].$$

This means that, putting $m_\varphi(h) := i\Lambda F_h^0 + \frac{1}{2}(\varphi \bar{\varphi}^h)_0$, we always have

$$\frac{d}{dt}\bigg|_{t=0} M_\varphi(he^{ts}) = 2 \int_X \text{Tr}[s m_\varphi(he^{ts})],$$

so that solving the projective vortex equation is equivalent to finding a critical point of the functional M_φ (compare with Lemma 3.3 [B2]).

Claim 6.3 *Suppose (\mathcal{E}, φ) is simple. Choose $B > 0$ and put*

$$\mathcal{Met}(l)_2^p(B) := \{h \in \mathcal{Met}(l)_2^p \mid \|m_\varphi(h)\|_{L^p} \leq B\} .$$

Then any $h \in \mathcal{Met}(l)_2^p(B)$ which minimizes M_φ on $\mathcal{Met}(l)_2^p(B)$ is a weak solution of the projective vortex equation.

The essential point is the injectivity of the operator $s \mapsto \Delta'_h(s) + \frac{1}{2}[(\varphi\bar{\varphi})s]_0$ acting on $L_2^p S_0(h)$. But from

$$\left\langle \Delta'_h s + \frac{1}{2}[\varphi\bar{\varphi}^h]s, s \right\rangle_h = \|\bar{\partial}_h(s)\|_h^2 + \|s\varphi\|_h^2$$

we see that this operator is injective on trace-free endomorphisms if (\mathcal{E}, φ) is simple. ■

Now we can follow again Bradlow's proof : if \mathcal{E} is φ -stable, then there exist positive constants C_1, C_2 such that for all $s \in L_2^p S_0(k)$ with $ke^s \in \mathcal{Met}(l)_2^p(B)$ the following "main estimate" holds:

$$\sup |s| \leq C_1 M_\varphi(ke^s) + C_2 .$$

This follows by applying Proposition 3.2 of [B2] to an arbitrary $\tau \in \mathbb{R}$ with

$$\max \left(\mu_g(\mathcal{E}), \sup_{\mathcal{F}' \in \mathcal{S}(\mathcal{E})} \mu_g(\mathcal{F}') \right) < \frac{(n-1)! \tau \text{Vol}_g(X)}{4\pi} < \inf_{\mathcal{F} \in \mathcal{S}_\varphi(\mathcal{E})} \mu_g(\mathcal{E}/\mathcal{F}) ,$$

since Bradlow's functional $\mathcal{M}_{\varphi, \tau}$ coincides on $\mathcal{Met}(l)$ with M_φ .

It remains to be shown that the existence of this main estimate implies the existence of a solution of the projective vortex equation.

The main estimate implies that for any $c > 0$, the set

$$\{s \in L_2^p S_0(k) \mid ke^s \in \mathcal{Met}(l)_2^p(B), \ M_\varphi(ke^s) < c\}$$

is bounded in L_2^p . Let (s_i) be a sequence in $L_2^p S_0(k)$ such that $ke^{s_i} \in \mathcal{Met}(l)_2^p(B)$ is a minimizing sequence for M_φ , and let s be weak limit. Then $h := ke^s$ is a weak solution of the projective vortex equation, which is smooth by elliptic regularity [B2].

Finally, we have to treat

Case 2: $\varphi \neq 0$, $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, with $\varphi \in H^0(\mathcal{E}')$, and the pair (\mathcal{E}', φ) is $\mu_g(E)$ -stable.

We wish to find metrics h' and h'' in \mathcal{E}' and \mathcal{E}'' , such that for $s := i\Lambda F_l$ the following equations are satisfied:

$$\begin{cases} h' \cdot h'' &= l \\ i\Lambda F_{h'} + \frac{1}{4}(\varphi \bar{\varphi}^{h'}) &= \frac{1}{2} \text{id}_{\mathcal{E}'} \\ i\Lambda F_{h''} &= \frac{1}{2}(s + \frac{1}{2}|\varphi|_{h'}^2) \text{id}_{\mathcal{E}''} . \end{cases}$$

Since the pair $(\mathcal{E}', \frac{1}{\sqrt{2}}\varphi)$ is $\mu_g(E)$ -stable, there exists by [OT1] a unique Hermitian metric h' in \mathcal{E}' solving the second of these equations. With this solution the third equation can be rewritten as

$$i\Lambda_g F_{h''} = s - i\Lambda_g F_{h'} .$$

Since $\int_X (s - i\Lambda_g F_{h'}) = \deg(\mathcal{E}'')$, we can solve this weak Hermitian-Einstein equation by a metric h'' , which is unique up to constant rescaling. The product $h' \cdot h''$ is a metric in $\mathcal{E}' \otimes \mathcal{E}'' = \mathcal{L}$ which has the same mean curvature s as l , and therefore differs from l by a constant factor. We can now simply rescale h'' by the inverse of this constant, and we get a pair of metrics satisfying the three equations above. ■

7 Moduli spaces of pairs

Let E be a differentiable vector bundle of rank r over a Kähler manifold (X, g) , and let \mathcal{L} be a holomorphic line bundle whose underlying differentiable bundle is $L := \det E$.

Proposition 7.1 *There exists a possibly non-Hausdorff complex analytic orbifold $\mathcal{M}^s(E, \mathcal{L})$ parametrizing isomorphism classes of simple holomorphic pairs of type (E, \mathcal{L}) . The open subset $\mathcal{M}^{ss}(E, \mathcal{L}) \subset \mathcal{M}^s(E, \mathcal{L})$ consisting of strongly simple pairs is a complex analytic space, and the points in $\mathcal{M}^s(E, \mathcal{L}) \setminus \mathcal{M}^{ss}(E, \mathcal{L})$ have neighbourhoods modeled on \mathbb{Z}/r -quotients.*

Proof: Since we use the same method as in the proof of Proposition 3.9, we only sketch the main ideas.

Let $\bar{\lambda}$ be the semiconnection defining the holomorphic structure of \mathcal{L} , and put $\bar{\mathcal{A}} := \bar{\mathcal{A}}_{\bar{\lambda}}(E) \times A^0(E)$, where $\bar{\mathcal{A}}_{\bar{\lambda}}(E)$ denotes the affine space of semiconnections in E inducing $\bar{\lambda}$ in $L = \det E$. The complex gauge group $SL(E)$ acts on $\bar{\mathcal{A}}$, and we write $\bar{\mathcal{A}}^s$ ($\bar{\mathcal{A}}^{ss}$) for the open subset of pairs whose stabilizer is contained in the center \mathbb{Z}/r of $SL(E)$ (is trivial). After suitable Sobolev completions, $\bar{\mathcal{A}}^{ss}$ becomes the total space of a holomorphic Hilbert principal $SL(E)$ -bundle over $\bar{\mathcal{B}}^{ss} := \bar{\mathcal{A}}^{ss}/SL(E)$.

A point $(\bar{\delta}, \varphi) \in \bar{\mathcal{A}}$ defines a pair of type (E, \mathcal{L}) iff it is integrable, i.e. iff it satisfies the following equations:

$$\begin{cases} F_{\bar{\delta}}^{02} &= 0 \\ \bar{\delta}\varphi &= 0. \end{cases} \quad (10)$$

Here $F_{\bar{\delta}}^{02} := \bar{\delta}^2$ is a $(0, 2)$ -form with values in the bundle $\text{End}_0(E)$ of trace-free endomorphisms. Moreover, isomorphism of pairs of type (E, \mathcal{L}) corresponds to equivalence modulo the action of the complex gauge group $SL(E)$.

Let $\bar{\sigma}$ be the map $\bar{\mathcal{A}} \longrightarrow A^{02}(\text{End}_0(E)) \oplus A^{01}(E)$ sending a pair $(\bar{\delta}, \varphi)$ to the left hand sides of (10). We endow the sets $\mathcal{M}_X^{ss}(E, \mathcal{L}) = Z(\sigma) \cap \bar{\mathcal{A}}^{ss}/SL(E)$ ($\mathcal{M}_X^s(E, \mathcal{L}) = Z(\sigma) \cap \bar{\mathcal{A}}^s/SL(E)$) with the structure of a complex analytic space (orbifold) as follows:

$\mathcal{M}_X^{ss}(E, \mathcal{L})$ is defined to be the vanishing locus of the section $\tilde{\sigma}$ in the Hilbert vector bundle $\bar{\mathcal{A}}^{ss} \times_{SL(E)} (A^{02}\text{End}_0(E) \oplus A^{01}E)$ over $\bar{\mathcal{B}}^{ss}$ which is defined by $\bar{\sigma}$.

To define the orbifold structure in $\mathcal{M}_X^s(E, \mathcal{L})$, we use local models derived from a deformation complex:

Let $\bar{p} = (\bar{\delta}, \varphi) \in \bar{\mathcal{A}}$ an integrable point. The associated deformation complex $\bar{\mathcal{D}}_{\bar{p}}$ is the cone over the evaluation map ev_{φ}^* :

$$ev_{\varphi}^q : A^{0q}(\text{End}_0(E)) \longrightarrow A^{0q}(E),$$

and has the form

$$\begin{aligned} 0 \longrightarrow A^0(\text{End}_0(E)) &\xrightarrow{\bar{D}_{\bar{p}}^0} A^{01}(\text{End}_0(E)) \oplus A^0(E) \xrightarrow{\bar{D}_{\bar{p}}^1} \\ &\xrightarrow{\bar{D}_{\bar{p}}^1} A^{02}(\text{End}_0(E)) \oplus A^{01}(E) \xrightarrow{\bar{D}_{\bar{p}}^2} \dots \end{aligned} \quad (\bar{\mathcal{D}}_{\bar{p}})$$

(compare with [OT1] §4). We define

$$\bar{S}_{\bar{p}}(\varepsilon) := \{\bar{p} + \beta | \beta \in A^{01}\text{End}_0(E) \oplus A^0 E, \bar{D}_{\bar{p}}^0 \bar{D}_{\bar{p}}^{0*}(\beta) + \bar{D}_{\bar{p}}^{1*}(\bar{\sigma}(\bar{p} + \beta)) = 0, \|\beta\| < \varepsilon\}.$$

The same arguments as in the proof of Proposition 3.9 show that for sufficiently small $\varepsilon > 0$, $\bar{S}_{\bar{p}}(\varepsilon)$ is a submanifold of $\bar{\mathcal{A}}$, whose tangent space in \bar{p} coincides with the first harmonic space $\bar{\mathbb{H}}_{\bar{p}}^1$ of the elliptic complex $(\bar{\mathcal{D}}_{\bar{p}})$. Therefore, we get a local finite dimensional model $Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})$ for the intersection $Z(\bar{\sigma}) \cap \bar{N}_{\bar{p}}(\varepsilon)$ of the integrable locus with the standard slice

$$\bar{N}_{\bar{p}}(\varepsilon) := \{\bar{p} + \beta | \beta \in A^{01}(\text{End}_0(E)) \oplus A^0(E), \bar{D}_{\bar{p}}^{0*}(\beta) = 0, \|\beta\| < \varepsilon\}$$

through \bar{p} . The restriction

$$\bar{\pi}|_{Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})} : Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)}) \longrightarrow \mathcal{M}_X^s(E, \mathcal{L})$$

of the orbit map is étale if $[\bar{p}] \in \mathcal{M}_X^{ss}(E, \mathcal{L})$, and induces an open injection

$$Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})/\mathbb{Z}/r \longrightarrow \mathcal{M}_X^s(E, \mathcal{L})$$

if $[\bar{p}] \in \mathcal{M}_X^s(E, \mathcal{L}) \setminus \mathcal{M}_X^{ss}(E, \mathcal{L})$. We define the orbifold structure of $\mathcal{M}_X^s(E, \mathcal{L})$ by taking the maps $\bar{\pi}|_{Z(\bar{\sigma}|_{\bar{S}_{\bar{p}}(\varepsilon)})}$ as orbifold-charts. \blacksquare

Our next purpose is to compare the two types of moduli spaces constructed in this paper. Let (X, g) be a Kähler surface endowed with the canonical $Spin^c$ -structure \mathfrak{c} . Let E be a $U(2)$ bundle with $\det E = K_X$, and denote by $\mathcal{M}^*(E)$ the moduli space of irreducible quaternionic monopoles associated to the $Spin^h(4)$ -structure defined by (\mathfrak{c}, E^\vee) (Lemma 1.3)

It follows from Proposition 4.1 that $\mathcal{M}^*(E)$ has a decomposition

$$\mathcal{M}^*(E) = \mathcal{M}^*(E)_{\alpha=0} \cup \mathcal{M}^*(E)_{\varphi=0},$$

where $\mathcal{M}^*(E)_{\alpha=0}$ ($\mathcal{M}^*(E)_{\varphi=0}$) is the Zariski closed subspace of $\mathcal{M}^*(E)$ cut out by the equation $\alpha = 0$ ($\varphi = 0$). The intersection

$$\mathcal{M}^*(E)_{\alpha=0} \cap \mathcal{M}^*(E)_{\varphi=0}$$

is the Donaldson instanton moduli space \mathcal{D}^* of irreducible projectively anti-self-dual connections in E , inducing the Chern connection in \mathcal{K}_X .

Proposition 7.2 *The affine isomorphism $\mathcal{A} \ni (C, \varphi) \mapsto (\bar{\partial}_C, \varphi) \in \bar{\mathcal{A}}$ induces a natural real analytic open embedding*

$$J : \mathcal{M}^*(E)_{\alpha=0} \hookrightarrow \mathcal{M}^s(E, \mathcal{K}_X)$$

whose image is the suborbifold of stable pairs of type (E, \mathcal{K}_X) .

Proof: Standard arguments (cf. [OT1]) show that J is an étale map which induces natural identifications of the local models.

A point $[(\bar{\delta}, \varphi)]$ lies in the image of J iff the $SL(E)$ -orbit of $(\bar{\delta}, \varphi)$ intersects the zero locus of the map

$$m : \bar{\mathcal{A}} \longrightarrow A^0(\mathfrak{su}(E)), \quad (\bar{\partial}_C, \varphi) \mapsto \Lambda_g F_C^0 - \frac{1}{2}(\varphi \bar{\varphi})_0.$$

Let (\mathcal{E}, φ) be the holomorphic pair of type (E, \mathcal{K}_X) defined by $(\bar{\delta}, \varphi)$. We can reformulate the condition above in the following way: $[(\mathcal{E}, \varphi)]$ lies in the image of J iff there exists a Hermitian metric h in \mathcal{E} inducing the Kähler metric in $\mathcal{K}_X = \det \mathcal{E}$ which satisfies the projective vortex equation (V). But we know already that this holds iff (\mathcal{E}, φ) is stable. Moreover, the unicity of the solution of the projective vortex equation is equivalent to the injectivity of J . ■

Using the remark after Proposition 4.1, we can now state the main result of this paper:

Theorem 7.3 *Let (X, g) be a Kähler surface with canonical bundle \mathcal{K}_X , and let E be a $U(2)$ -bundle with $\det E = \mathcal{K}_X$. Consider the Spin^h -structure associated with the canonical $\text{Spin}^c(4)$ -structure and the $U(2)$ -bundle E^\vee . The corresponding moduli space of irreducible quaternionic monopoles is a union of two Zariski closed subspaces. Each of these subspaces is naturally isomorphic as a real analytic orbifold to the moduli space of stable pairs of type (E, \mathcal{K}_X) . There exists a real analytic involution on the quaternionic moduli space which interchanges these two closed subspaces. The fixed point set of this involution is the Donaldson moduli space of instantons in E with fixed determinant, modulo the gauge group $SU(E)$. The closure of the complement of the Donaldson moduli space intersects the moduli space of instantons in the Brill-Noether locus.*

The union \mathcal{M}^{SW} of all rank 1-Seiberg-Witten moduli spaces associated with splittings $E = E' \oplus E''$ corresponds to the subspace of stable pairs of type ii).

References

- [B1] Bradlow, S. B.: *Vortices in holomorphic line bundles over closed Kähler manifolds*, Comm. Math. Phys. 135, 1-17 (1990)
- [B2] Bradlow, S. B.: *Special metrics and stability for holomorphic bundles with global sections*, J. Diff. Geom. 33, 169-214 (1991)
- [D] Donaldson, S.: *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 3, 1-26 (1985)
- [DK] Donaldson, S.; Kronheimer, P.B.: *The Geometry of four-manifolds*, Oxford Science Publications (1990)
- [FU] Freed D. S. ; Uhlenbeck, K.: *Instantons and Four-Manifolds*. Springer-Verlag 1984.
- [HH] Hirzebruch, F.; Hopf H.: *Felder von Flächenelementen in 4-dimensionalen 4-Mannigfaltigkeiten*, Math. Ann. 136 (1958)
- [H] Hitchin, N.: *Harmonic spinors*, Adv. in Math. 14, 1-55 (1974)
- [JPW] Jost, J.; Peng, X.; Wang, G. : *Variational aspects of the Seiberg-Witten functional*, Preprint, dg-ga/9504003, April (1995)
- [K] Kobayashi, S.: *Differential geometry of complex vector bundles*, Princeton University Press (1987)
- [KM] Kronheimer, P.; Mrowka, T.: *The genus of embedded surfaces in the projective plane*, Preprint (1994)
- [LM] Labastida, J. M. F.; Marino M.: *Non-abelian monopoles on four manifolds*, Preprint, Departamento de Física de Partículas, Santiago de Compostela, April (1995)
- [LT] Lübke, M.; Teleman, A.: *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co, to appear.
- [M] Miyajima K.: *Kuranishi families of vector bundles and algebraic description of the moduli space of Einstein-Hermitian connections*, Publ. R.I.M.S. Kyoto Univ. 25, 301-320 (1989)
- [OSS] Okonek, Ch.; Schneider, M.; Spindler, H: *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser, Boston (1980)
- [OT1] Okonek, Ch.; Teleman A.: *The Coupled Seiberg-Witten Equations, Vortices, and Moduli Spaces of Stable Pairs*, Preprint, Zürich, Jan. 13-th, (1995)

- [OT2] Okonek, Ch.; Teleman A.: *Les invariants de Seiberg-Witten et la conjecture de Van De Ven*, to appear in Comptes Rendus
- [OT3] Okonek, Ch.; Teleman A.: *Seiberg-Witten invariants and rationality of complex surfaces*, Preprint, Zürich, March (1995)
- [W] Witten, E.: *Monopoles and four-manifolds*, Mathematical Research Letters 1, 769-796 (1994)

Authors addresses:

Mathematisches Institut, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich
e-mail: okonek@math.unizh.ch ; teleman@math.unizh.ch